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FIXED POINT THEOREM IN MENGER SPACE FOR WEAKLY SEMI-COMPATIBLE MAPPINGS

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ABSTRACT

In this paper, the concept of weakly semi-compatibility and weak compatibility in Menger space has been applied to prove a commonly fixed point theorem for six self-maps. An example has also given to support the result.

KEYWORDS: *Probabilistic Metric Space, Menger Space, Common Fixed Point, Compatible Maps, Weakly Semi-Compatible Maps, Weak Compatibility. F Type (A), Fuzzy Metric Space, AMS Subject Classification: Primary 47H10, Secondary 54H25*

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INTRODUCTION

There have been a number of generalizations of metric space. One such generalization is Menger space initiated by Menger [4]. It is a probabilistic generalization in which we assign to any two points x and y, a distribution function $F_{x,y}$. Schweizer and Sklar [8] studied this concept and gave some fundamental results on this space. Sehgal and Bharucha-Reid [9] obtained a generalization of Banach Contraction Principle on a complete Menger space which is a milestone in developing fixed-point theory in Menger space.

Recently, Jungck and Rhoades [3] termed a pair of self-maps to be coincidentally commuting or equivalently weakly compatible if they commute at their coincidence points. Sessa [10] initiated the tradition of improving commutativity in fixed-point theorems by introducing the notion of weak commuting maps in metric spaces. Jungck [2] soon enlarged this concept to compatible maps. The notion of compatible mapping in a Menger space has been introduced by Mishra [5].

Cho, Sharma, and Sahu [1] introduced the concept of semi-compatibility in ad-complete topological space. Popa [7] proved interesting fixed point results using implicit real functions and semi-compatibility in d-complete topological space. In the sequel, Pathak and Verma [6] proved a commonly fixed point theorem in Menger space using compatibility and weak compatibility.

In this paper, a fixed point theorem for six self-maps has been proved using the concept of weakly semi-compatible maps and weak compatible maps.

Preliminaries

Definition: A mapping $F: R \to R^{+}$ is called a distribution if it is non-decreasing left continuous with

inf { F (t) | $t \in R$ } = 0 and sup { F (t) | $t \in R$} = 1.

We shall denote by L the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & t \le 0 \\ 1, & t > 0 \end{cases}.$$

Definition: A triangular norm * (shortly t-norm) is a binary operation on the unit interval [0, 1] such that for all a, b, c, d □□ [0, 1] the following conditions are satisfied:

(a) a * 1 = a;

((b) a * b = b * a;

((c) a * b □ c * d whenever a □ c and b □ d;

(d) (d) a * (b * c) = (a * b) * c.

Examples of t-norms are a * b = max {a + b - 1, 0} and a * b = min {a, b}.

Definition: **[8]** A probabilistic metric space (PM-space) is an ordered pair (X, F) consisting of a non empty set X and a function F: X × X □□L, where L is the collection of all distribution functions and the value of F at (u, v) □□X × X is represented by $F_{u,v}$. The function $F_{u,v}$ assumed to satisfy the following conditions:

(PM-1) $F_{u,v}(x) = 1$, for all x > 0, if and only if u = v;

(PM-2) $F_{u,v}(0) = 0$;

(PM-3) $F_{u,v} = F_{v,u}$;

(PM-4) If $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$ then $F_{u,w}(x + y) = 1$,

for all u,v,w · X and x, y > 0.

Definition: **[8]** A Menger space is a triplet (X, F, t) where (X, F) is a PM-space and * is a t-norm such that the inequality

(PM-5) $F_{u,w}(x + y)$ · $F_{u,v}(x) * F_{v,w}(y)$, for all u, v, w · ·X, x, y · 0.

Proposition: **[9]** Let (X, d) be a metric space. Then the metric d induces a distribution function F defined by $F_{x,y}$ {t} = H(t - d(x,y)) for all x, y · X and t > 0. If t-norm * is a * b = min {a, b} for all a, b · [0, 1] then (X, F, *) is a Menger space. Further, (X, F, *) is a complete Menger space if (X, d) is complete.

Definition: [5] Let (X, F, *) be a Menger space and * be a continuous t-norm.

- A sequence $\{x_n\}$ in X is said to be converge to a point x in S (written x_n □ x) iff for every □ > 0 and □ □ (0,1), there exists an integer $n_0=n_0$ (□, □) such that $F_{x_n, x}$ (□) > 1 - □ for all n □ n_0.
- A sequence $\{x_n\}$ in X is said to be Cauchy if for every □ >0 and □ □ (0,1), there exists an integer $n_0 = n_0$(□, □) such that $F_{x_n, x_{n+p}}$ (□) >1 - □ for all n □ n_0 and p > 0.
- A Menger space in which every Cauchy sequence is convergent is said to be complete.

Remark: If * is a continuous t-norm, it follows from (PM-4) that the limit of a sequence in Menger space is uniquely determined.

Definition: [11] Self-mappings A and S of a Menger space (X, F, t) are said to be weak compatible if they commute at their coincidence points i.e. Ax = Sx for x □ X implies ASx = SAx.

Definition: [5] Self mappings A and S of a Menger space (X, F, t) are said to be compatible if F_{ASx_n, SAx_n} (x) □ 1 for all x > 0, whenever $\{x_n\}$ is a sequence in X such that Ax_n, Sx_n □ u for some u in X, as n □□□.

Definition: Self mappings A and S of a Menger space (X, F, t) are said to be weakly semi-compatible if $F_{ASx_n, Su}$ (x) □ 1 or $F_{SAx_n, Au}$ (x) □ 1 for all x > 0, whenever $\{x_n\}$ is a sequence in X such that Ax_n, Sx_n □ u, for some u in X, as n □□□.

Now, we give an example of a pair of self-maps (S, T) which is weakly semi-compatible but not compatible. Further, we observe here that the pair (T, S) is not weakly semi-compatible though (S, T) is weakly semi-compatible.

Example: Let (X, d) be a metric space where X = [0, 1] and (X, F, t) be the induced Menger space with $F_{p,q}$(□) = H(□ - d(p, q)), □ p, q □ X and □ □ > 0. Define self maps S and T as follows:

DSMT4 $$Sx = \begin{cases} x & \text{if} \quad 0 \le x < \frac{1}{2} \\ 1 & \text{if} \quad \frac{1}{2} \le x \le 1 \end{cases}$$ And DSMT4 $$Tx = \begin{cases} 1-x & \text{if} \quad 0 \le x < \frac{1}{2} \\ 1 & \text{if} \quad \frac{1}{2} \le x \le 1 \end{cases}$$

Take x_n = DSMT4 $\frac{1}{2} - \frac{1}{n}$. Now,

$F_{Sx_n, 1/2}$ (□) = H(□ - (1/n)).

Therefore, DSMT4 $\lim_{n\to\infty} F_{Sx_n, 1/2}$ (□) = H(□) = 1.

Hence, Sx_n □ 1/2 as n □□□

Similarly, Tx_n □ 1/2 as n □□□

Also

F_{STx_n,TSx_n}(□) = DSMT4 $H\left(\varepsilon - \left(\frac{1}{2} - \frac{1}{n}\right)\right)$ □ 1, □□□ > 0.

Hence, the pair (S, T) is not compatible.

Again, DSMT4 $\lim_{n\to\infty}$ $F_{STx_n,Tx}$(□) = DSMT4 $\lim_{n\to\infty}$ $F_{STx_n,1}$(□) = H(□ - |1-1|) = 1 □ □ > 0.

Thus, (S, T) is weakly semi-compatible.

Now, DSMT4 $\lim_{n\to\infty}$ $F_{TSx_n,Sx}$(e) □ 1, □ □ > 0.

Thus, (T, S) is not weakly semi-compatible

Remark: In view of the above example, it follows that the concept of weakly semi-compatibility is more general than that of compatibility.

Lemma: [11] Let $\{x_n\}$ be a sequence in a Menger space (X, F, *) with continuous t-norm * and t * t □ t. If there exists a constant k □ (0, 1) such that

$F_{x_n, x_{n+1}}(kt)$ □ $F_{x_{n-1}, x_n}(t)$

for all t > 0 and n = 1, 2, 3,..., then $\{x_n\}$ is a Cauchy sequence in X.

Main Result

Theorem: Let A, B, S, T, L and M be self maps of a complete Menger space (X, F, *) with t* t □ t satisfying :

L(X) □ ST(X), M(X) □ AB(X);

AB = BA, ST = TS, LB = BL, MT = TM;

either L or AB is continuous;

(L, AB) is weakly semi-compatible and (M, ST) is weak compatible; There exists a constant k □ (0, 1) such that

$F^2_{Lx,My}(kt)*[F_{ABx,Lx}(kt).F_{STy,My}(kt)]$ □ $[pF_{ABx,\,Lx}(t) + qF_{ABx,\,STy}(t)].F_{ABx,\,My}(2kt)$

for all x, y □ X and t > 0 where 0 < p, q < 1 such that p + q = 1.

Then A, B, S, T, L, and M have a unique common fixed point in X.

Proof: Suppose x_0 □ X. From condition (3.1.1) □ x_1, x_2 □ X such that

$Lx_0 = STx_1$ and $Mx_1 = ABx_2$.

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$y_{2n} = Lx_{2n} = STx_{2n+1}$ and $y_{2n+1} = Mx_{2n+1} = ABx_{2n+2}$ for n = 0, 1, 2,

Step 1: Taking $x = x_{2n}$ and $y = x_{2n+1}$ in (3.1.5), we have

$F^2_{Lx_{2n},Mx_{2n+1}}(kt)*[F_{ABx_{2n},Lx_{2n}}(kt).F_{STx_{2n+1}, Mx_{2n+1}}(kt)]$

$\square$ $[pF_{ABx_{2n}, Lx_{2n}}(t) + qF_{ABx_{2n}, STx_{2n+1}}(t)].F_{ABx_{2n}, Mx_{2n+1}}(2kt)$

$F^2_{y_{2n}, y_{2n+1}}(kt)*[F_{y_{2n-1},y_{2n}}(kt).F_{y_{2n}, y_{2n+1}}(kt)]$

$\square$ $[pF_{y_{2n}, y_{2n-1}}(t) + qF_{y_{2n-1}, y_{2n}}(t)].F_{y_{2n}, y_{2n+1}}(2kt)$

$F_{y_{2n}, y_{2n+1}}(kt)[F_{y_{2n-1},y_{2n}}(kt) * F_{y_{2n}, y_{2n+1}}(kt)]$

$\square$ $(p + q)F_{y_{2n}, y_{2n-1}}(t).F_{y_{2n-1}, y_{2n+1}}(2kt)$

$F_{y_{2n},y_{2n+1}}(kt)F_{y_{2n-1}, y_{2n+1}}(2kt)$ $\square$ $Fy_{2n-1, y_{2n}}(t)Fy_{2n-1, y_{2n+1}}(2kt)$.

Hence, we have

$F_{y_{2n}, y_{2n+1}}(kt)$ $\square$ $F_{y_{2n-1}, y_{2n}}(t)$.

Similarly, we also have

$F_{y_{2n+1}, y_{2n+2}}(kt)$ $\square$ $F_{y_{2n}, y_{2n+1}}(t)$.

In general, for all n even or odd, we have

$F_{y_n, y_{n+1}}(kt)$ $\square$ $F_{y_{n-1}, y_n}(t)$

for k $\square$ (0, 1) and all t > 0. Thus, by lemma 2.1, $\{y_n\}$ is a Cauchy sequence in X. Since (X, F, *) is complete, it converges to a point z in X. Also its subsequences converge as follows :

$\{Lx_{2n}\}$ $\square$ z, $\{ABx_{2n}\}$ $\square$ z, $\{Mx_{2n+1}\}$ $\square$ z and $\{STx_{2n+1}\}$ $\square$ z. (3.1.6)

Case I: Suppose AB is Continuous

As AB is continuous and (L, AB) is weakly semi-compatible, we get

$LABx_{2n+2}$ $\square$ Lz and $LABx_{2n+2}$ $\square$ ABz. (3.1.7)

Since the limit in Menger space is unique, we get

$Lz = ABz.$ (3.1.8)

Step 2: By taking $x = ABx_{2n}$ and $y = x_{2n+1}$ in (3.1.5), we have

$F^2_{LABx_{2n},Mx_{2n+1}}(kt)*[F_{ABABx_{2n},LABx_{2n}}(kt).F_{STx_{2n+1},Mx_{2n+1}}(kt)]$

$\square\square\ [pF_{ABABx_{2n},LABx_{2n}}(t) + qF_{ABABx_{2n},STx_{2n+1}}(t)].F_{ABABx_{2n},Mx_{2n+1}}(2kt).$

Taking limit $n \square\square\square$

$F^2_{z,ABz}(kt)*[F_{ABz,ABz}(kt).F_{z,z}(kt)] \square [pF_{ABz,ABz}(t) + qF_{z,ABz}(t)].F_{z,ABz}(2kt)$

$\square\ [p + qF_{z,ABz}(t)]F_{z,ABz}(kt)]$

$F_{z,ABz}(kt) \square\ p + qF_{z,ABz}(t)$

$\square\ p + qF_{z,ABz}(kt)$

$F_{z,ABz}(kt) \square \ \text{DSMT4}\ \dfrac{p}{1-q} = 1$

for $k \square (0, 1)$ and all $t > 0$. Thus, we have

$z = ABz.$

Step 3: By taking $x = z$ and $y = x_{2n+1}$ in (3.1.5), we have

$F^2_{Lz,Mx_{2n+1}}(kt)*[F_{ABz,Lz}(kt).F_{STx_{2n+1},Mx_{2n+1}}(kt)]$

$\square\ [pF_{ABz,Lz}(t) + qF_{ABz,STx_{2n+1}}(t)].F_{ABz,Mx_{2n+1}}(2kt)$

Taking limit $n \square\square\square$

$F^2_{z,Lz}(kt)*[F_{z,Lz}(kt).F_{z,z}(kt)] \square [pF_{z,Lz}(t) + qF_{z,z}(t)].F_{z,z}(2kt)$

$F^2_{z,Lz}(kt)*F_{z,Lz}(kt) \square\ pF_{z,Lz}(t) + q.$

Noting that $F^2_{z,Lz}(kt) \square 1$ and using (c) in Definition 2.2, we have

$F_{z,Lz}(kt) \square\ pF_{z,Lz}(t) + q$

$\square\ pF_{z,Lz}(kt) + q$

$F_{z,Lz}(kt) \square \ \text{DSMT4}\ \dfrac{q}{1-p} = 1$

for $k \square (0, 1)$ and all $t > 0$. Thus, we have $z = Lz = ABz$.

Step 4: By taking x = Bz and y = x_{2n+1} in (3.1.5), we have

$F^2_{LBz,Mx_{2n+1}}(kt)*[F_{ABBz,LBz}(kt).F_{STx_{2n+1},\, Mx_{2n+1}}(kt)]$

$\square$ $[pF_{ABBz,\, LBz}(t) + qF_{ABBz,\, STx_{2n+1}}(t)].F_{ABBz,\, Mx_{2n+1}}(2kt)$.

Since AB = BA and BL = LB, we have

L(Bz) = B(Lz) = Bz and

AB(Bz) = B(ABz) = Bz.

Taking limit n $\square\square\square$, we have

$F^2_{z,Bz}(kt)*[F_{Bz,Bz}(kt).F_{z,\, z}(kt)]$ $\square$ $[pF_{Bz,\, Bz}(t) + qF_{z,\, Bz}(t)].F_{z,\, Bz}(2kt)$

$F^2_{z,Bz}(kt)$ $\square$ $[p + qF_{z,\, Bz}(t)]F_{z,\, Bz}(2kt)$

$\square$ $[p + qF_{z,\, Bz}(t)]F_{z,\, Bz}(kt)$

$F_{z,Bz}(kt)$ $\square$ $p + qF_{z,\, Bz}(t)$

$\square$ $p + qF_{z,\, Bz}(kt)$

$F_{z,Bz}(kt)$ $\square\square$ DSMT4 $\dfrac{p}{1-q} = 1$

for k $\square$ (0, 1) and all t > 0.

Thus, we have

z = Bz.

Since z = ABz, we also have

z = Az.

Therefore, z = Az = Bz = Lz.

Step 5: Since L(X) $\square\square\square$ST(X) there exists v $\square$ X such that

z = Lz = STv.

By taking x = x_{2n} and y = v in (3.1.5), we get

$F^2_{Lx_{2n},Mv}(kt)*[F_{ABx_{2n},Lx_{2n}}(kt).F_{STv,\, Mv}(kt)]$

$\geq [pF_{ABx_{2n}, Lx_{2n}}(t) + qF_{ABx_{2n}, STv}(t)].F_{ABx_{2n}, Mv}(2kt)$.

Taking limit as $n \to \infty$, we have

$F^2_{z,Mv}(kt)*[F_{z,z}(kt).F_{z, Mv}(kt)] \geq [pF_{z, z}(t) + qF_{z, z}(t)].F_{z, Mv}(2kt)$

$F^2_{z,Mv}(kt)*F_{z,Mv}(kt) \geq (p + q)F_{z, Mv}(2kt)$.

Noting that $F^2_{z, Mv}(kt) \leq 1$ and using (c) in Definition 2.2, we have

$F_{z, Mv}(kt) \geq F_{z, Mv}(2kt)$

$\geq F_{z, Mv}(t)$.

Thus, we have

z = Mv and so z = Mv = STv.

Since (M, ST) is weakly compatible, we have

STMv = MSTv.

Thus, STz = Mz.

Step 6: By taking $x = x_{2n}$, $y = z$ in (3.1.5) and using Step 5, we have

$F^2_{Lx_{2n},Mz}(kt)*[F_{ABx_{2n},Lx_{2n}}(kt).F_{STz, Mz}(kt)]$

$\geq [pF_{ABx_{2n}, Lx_{2n}}(t) + qF_{ABx_{2n}, STz}(t)].F_{ABx_{2n}, Mz}(2kt)$

which implies that, as $n \to \infty$

$F^2_{z,Mz}(kt)*[F_{z,z}(kt).F_{Mz, Mz}(kt)] \geq [pF_{z, z}(t) + qF_{z, Mz}(t)].F_{z, Mz}(2kt)$

$F^2_{z,Mz}(kt) \geq [p + qF_{z, Mz}(t)]F_{z, Mz}(2kt)$

$\geq [p + qF_{z, Mz}(t)]F_{z, Mz}(kt)$

$F_{z,Mz}(kt) \geq p + qF_{z, Mz}(t)$

$\geq p + qF_{z, Mz}(kt)$

$F_{z,Mz}(kt) \geq$ DSMT4 $\dfrac{p}{1-q} = 1$.

Thus, we have z = Mz and therefore z = Az = Bz = Lz = Mz = STz

Step 7: By taking $x = x_{2n}$, $y = Tz$ in (3.1.5), we have

$F^2_{Lx_{2n},MTz}(kt)*[F_{ABx_{2n},Lx_{2n}}(kt).F_{STTz,\ MTz}(kt)]$

□ $[pF_{ABx_{2n},\ Lx_{2n}}(t) + qF_{ABx_{2n},\ STTz}(t)].F_{ABx_{2n},\ MTz}(2kt)$.

Since MT = TM and ST = TS, we have

MTz = TMz = Tz and ST(Tz) = T(STz) = Tz.

Letting n □□□, we have

$F^2_{z,\ Tz}(kt)*[F_{z,\ z}(kt).F_{Tz,\ Tz}(kt)]$ □ $[pF_{z,\ z}(t) + qF_{z,\ Tz}(t)].F_{z,\ Tz}(2kt)$

$F_{z,Tz}(kt)$ □ $p + qF_{z,\ Tz}(t)$

□ $p + qF_{z,\ Tz}(kt)$

$F_{z,Tz}(kt)$ □□ DSMT4 $\dfrac{p}{1-q} = 1$.

Thus, we have z = Tz. Since Tz = STz, we also have z = Sz.

Therefore, z = Az = Bz = Lz = Mz = Sz = Tz, that is, z is the common fixed point of the six maps.

Case II: L is Continuous

Since L is continuous, LLx_{2n} □ Lz and $L(AB)x_{2n}$ □ Lz.

Since (L, AB) is weakly semi-compatible, $L(AB)x_{2n}$ □ ABz.

Step 8: By taking $x = Lx_{2n}$, $y = x_{2n+1}$ in (b), we have

$F^2_{LLx_{2n},Mx_{2n+1}}(kt)*[F_{ABLx_{2n},LLx_{2n}}(kt).F_{STx_{2n+1},\ Mx_{2n+1}}(kt)]$

□ $[pF_{ABLx_{2n},\ LLx_{2n}}(t) + qF_{ABLx_{2n},\ STx_{2n+1}}(t)].F_{ABLx_{2n},\ Mx_{2n+1}}(2kt)$

Letting n □□□, we have

$F^2_{z,\ Lz}(kt)*[F_{Lz,\ Lz}(kt).F_{z,\ z}(kt)]$ □ $[pF_{Lz,\ Lz}(t) + qF_{z,\ Lz}(t)].F_{z,\ Lz}(2kt)$

$F^2_{z,\ Lz}(kt)$ □ $[p + qF_{z,\ Lz}(t)]F_{z,\ Lz}(2kt)$

□ $[p + qF_{z,\ Lz}(t)]F_{z,\ Lz}(kt)$,

$F_{z,\ Lz}(kt)$ □ $p + qF_{z,\ Lz}(t)$

□ $p + qF_{z,\ Lz}(kt)$,

$F_{z, Lz}(kt)$ □ DSMT4 $\dfrac{p}{1-q}$ = 1.

Thus, we have z = Lz and using Steps 5-7, we have

z = Lz = Mz = Sz = Tz.

Step 9: Since M(X) □ AB(X), there exists v □ X such that

z = Mz = ABv.

By taking x = v, y = x_{2n+1} in (3.1.5), we have

$F^2_{Lv,Mx_{2n+1}}(kt)*[F_{ABv,Lv}(kt).F_{STx_{2n+1}, Mx_{2n+1}}(kt)]$

□ $[pF_{ABv, Lv}(t) + qF_{ABv, STx_{2n+1}}(t)].F_{ABv, Mx_{2n+1}1}(2kt)$.

Taking limit as n □□□, we have

$F^2_{z,Lv}(kt)*[F_{z,Lv}(kt).F_{z, z}(kt)]$ □ $[pF_{z, Lv}(t) + qF_{z, z}(t)].F_{z, z}(2kt)$

$F^2_{z,Lv}(kt)*F_{z,Lv}(kt)$ □ $pF_{z, Lv}(t) + q$

□ $pF_{z, Lv}(kt) + q$.

Noting that $F^2_{z, Lv}(kt)$ □ 1 and using (c) in Definition 2.2, we have

$F_{z,Mv}(kt)$ □ $pF_{z, Lv}(kt) + q$,

$F_{z,Mv}(kt)$ □ DSMT4 $\dfrac{q}{1-p}$ = 1.

Thus, we have z = Lv = ABv.

Since (L, AB) is weakly semi-compatible, we have

Lz = ABz and using Step 4, we also have z = Bz.

Therefore, z = Az = Bz = Sz = Tz = Lz = Mz, that is, z is the common fixed point of the six maps in this case also.

Step 10: For uniqueness, let w (w □ z) be another common fixed point of A, B, S, T, L and M.

Taking x = z, y = w in (3.1.5), we have

$F^2_{Lz,Mw}(kt)*[F_{ABz,Lz}(kt).F_{STw, Mw}(kt)]$ □ $[pF_{ABz, Lz}(t) + qF_{ABz, STw}(t)].F_{ABz, Mw}(2kt)$

which implies that

$F^2_{z,w}(kt) \square [p + qF_{z,w}(t)]F_{z,w}(2kt)$

$\square [p + qF_{z,w}(t)]F_{z,w}(kt)$,

$F_{z,w}(kt) \square p + qF_{z,w}(t)$

$\square p + qF_{z,w}(kt)$

$F_{z,w}(kt) \square$ DSMT4 $\dfrac{p}{1-q} = 1.$

Thus, we have z = w.

This completes the proof of the theorem.

If we take $B = T = I_X$ (the identity map on X) in theorem 3.1, we have the following:

Corollary: Let A, S, L and M be self maps of a complete Menger space (X, F, *) with $t * t \square t$ satisfying :

- $L(X) \square S(X)$, $M(X) \square A(X)$;
- Either L or A is continuous;
- (L, A) is weakly semi-compatible and (M, S) is weak compatible;
- there exists a constant $k \square (0, 1)$ such that

$F^2_{Lx,My}(kt)*[F_{Ax,Lx}(kt).F_{Sy,My}(kt)] \square [pF_{Ax,Lx}(t) + qF_{Ax,Sy}(t)].F_{Ax,My}(2kt)$ for all $x, y \square X$ and $t > 0$ where $0 < p, q < 1$ such that $p + q = 1$.

Then A, S, L, and M have a unique common fixed point in X.

REFERENCES

1. *Y.J. Cho, B.K. Sharma and R.D. Sahu, Semi-compatibility and fixed points, Math. Japon. 42 (1), (1995), 91-98.*
2. *[G. Jungck, Compatible mappings and common fixed points, Internat. J. Math. and Math. Sci. 9(4), (1986), 771-779.*
3. *G. Jungck and B.E. Rhoades, Fixed points for set valued functions without continuity, Indian J. Pure Appl. Math. 29(1998), 227-238.*
4. *K. Menger, Statistical metrics, Proc. Nat. Acad. Sci. USA. 28(1942), 535 -537.*
5. *S.N. Mishra, Common fixed points of compatible mappings in PM-spaces, Math. Japon. 36(2), (1991), 283-289.*

6. H.K. Pathak and R.K. Verma, Common fixed point theorems for weakly compatible mappings in Menger space and application, Int. Journal of Math. Analysis, Vol. 3, 2009, No. 24, 1199-1206.

7. V. Popa, Fixed points for non-surjective expansion mappings satisfying an implicit relation, Bul. Stiint. Univ. Baia Mare Ser. B Fasc. Mat.-Inform. 18 (1), (2002), 105–108.

8. B. Schweizer and A. Sklar, Statistical metric spaces, Pacific J. Math. 10 (1960), 313-334.

9. V.M. Sehgal and A.T. Bharucha-Reid, Fixed points of contraction maps on probabilistic metric spaces, Math. System Theory 6(1972), 97- 102.

10. S. Sessa, On a weak commutativity condition of mappings in fixed point consideration, Publ. Inst. Math. Beograd 32(46), (1982), 146-153.

11. B. Singh and S. Jain, A fixed point theorem in Menger space through weak compatibility, J. Math. Anal. Appl., 301 (2005), 433